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NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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[Continued from January Number.]

PROPOSITION XXXII. *Now I say there is (in hypothesis of acute angle) a certain determinate acute angle BAX drawn under which AX* (Fig. 33.) *only at an infinite distance meets BX, and thus is a limit in part from within, in part from without; on the one hand of all those which under lesser acute angles meet the aforesaid BX at a finite distance; on the other hand also of the others which under greater acute angles, even to a right angle inclusive, have a common perpendicular in two distinct points with BX.*

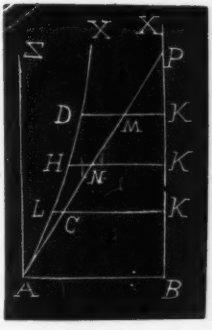


Fig. 33.

PROOF. First it holds (from Cor. II. after Proposition XXIX.) that no determinate acute angle will be the greatest of all drawn under which a straight from the point A meets the aforesaid BX at a finite distance

Secondly, it holds in like manner that (in hypothesis of acute angle) no acute angle will be the least of all drawn under which a straight has a common perpendicular in two distinct points with BX; since indeed (from what precedes) there can be no determinate limit, such that there cannot be found, under a lesser angle

constituted at the point A, a common perpendicular in two distinct points, which is less than any assignable length R.

And hence follows thirdly, that (in this hypothesis) there must be a certain determinate acute angle BAX, drawn under which AX so approaches ever more to BX, that only at an infinite distance does it meet it.

But further that this AX is a limit in part from within in part from without of each of the aforesaid classes of straights is proved thus. First, it agrees with those straights which meet BX at a finite distance since it also finally meets; but it differs, because it meets only at an infinite distance.

But secondly it also agrees, and at the same time differs from those straights which have a common perpendicular in two distinct points with BX; because it also has a common perpendicular with BX; but in one and the same point X infinitely distant. But this latter ought to be considered demonstrated in Proposition XXVIII., as I point out in its corollary.

Therefore it holds, that (in the hypothesis of acute angle) there will be a certain determinate acute angle BAX, drawn under which AX only at an infinite distance meets BX, and thus is a limit in part from within, in part from without; on the one hand of all those which under lesser acute angles meet the aforesaid BX at a finite distance; on the other hand also of the others which under greater acute angles, even to a right angle inclusive, have a common perpendicular in two distinct points with BX. Quod erat etc.

[To be Continued.]

ON THE BEST METHOD OF SOLVING THE MARKINGS OF JUDGES OF CONTESTS.

By F. R. MOULTON.

The fact that many different methods are in use for deciding from the markings of judges the relative standing of the participants in oratorical and similar contests, and that several different methods have been stated to be the best by Professors of Mathematics in some of our colleges, may be taken as the excuse for this paper. It is questionable whether such a problem can be solved by perfectly rigorous mathematical processes, but it has seemed that a method similar to that of Hagen in the theory of probability may be applied with advantage. Let us agree to adopt the following hypotheses.

(1) The judges mark each contestant independently and by the same scale.

(2) There is a true marking for each contestant as compared to a fixed ideal.

(3) The deviation of each judge's markings from these true markings, are the result of a very great many influences, such as, the inherited inclinations re-

sulting from the entire experience of the judge's ancestry, the judge's own experiences, his business or profession, health, etc., etc.

These influences will have different degrees of importance. Let us select a unit in terms of which all the influences acting upon every judge can be expressed. Then we shall consider that an influence whose importance is k times the unit influence is equivalent to k unit influences working together. With this understanding the whole number of influences is some large number, μ. The coefficients of those which do not operate in the case of particular judges are zero.

(4) In the long run the unit influence will be just as liable to cause the judge to err in one direction as the other. (This is plainly a pure assumption but is the only one that can be made.)

Let us now consider the question in two parts; first, an investigation of the law of distribution of errors, and second, the best way of eliminating their influence.

I. *Investigation of the expression for the probability of the existence of an error of a given magnitude.*

By hypothesis (4) the probability that a unit influence will give a positive error is $p=\frac{1}{2}$, and negative $q=\frac{1}{2}$.

Then all possible combinations of positive and negative errors is given by the terms of the expansion

$$(1) \qquad (p+q)^{\mu}=1=\Sigma\frac{\mu\,!}{m\,!\,n\,!}p^m q^n$$

where $m+n=\mu$. The general term is

$$(2) \qquad T_n=\frac{\mu\,!}{m\,!\,n\,!}p^m q^n$$

which is the probability of m positive and n negative errors. By Stirling's theorem

$$\mu\,!=\mu^{\mu}e^{-\mu}\sqrt{2\pi\mu}.$$

Therefore (2) becomes

$$(3) \qquad T_n=\left(\frac{\mu p}{m}\right)^m\left(\frac{\mu q}{n}\right)^n\frac{\sqrt{\mu}}{\sqrt{(2\pi\mu)}}=\left(\frac{\mu p}{m}\right)^{m+\frac{1}{2}}\left(\frac{\mu q}{n}\right)^{n+\frac{1}{2}}\frac{1}{\sqrt{(2\pi\mu pq)}}.$$

We now find what values of m and n make T_n a maximum. Changing n to $n-1$ and $n+1$ successively in (2) we have

$$(4) \qquad \begin{cases} T_{n-1}=\dfrac{\mu\,!}{(m+1)!\,(n-1)!}p^{m+1}q^{n-1}=T_n\cdot\dfrac{n}{m+1}\dfrac{p}{q} \\ T_{n+1}=\dfrac{\mu\,!}{(m-1)!\,(n+1)!}p^{m-1}q^{n+1}=T_n\cdot\dfrac{m}{n+1}\dfrac{q}{p} \end{cases}$$

If T_n is a maximum $T_n>T_{n\pm1}$. Therefore

$$(5)\qquad \begin{cases} 1 > \dfrac{n}{m+1}\dfrac{p}{q} \text{ and } 1 > \dfrac{m}{n+1}\dfrac{q}{p} \\ m+n=\mu \text{ and } q=1-p. \end{cases}$$

Eliminating n and q we have

$$1 > \frac{\mu-m}{m+1}\frac{p}{1-p} \text{ and } 1 > \frac{m}{\mu-m+1}\frac{1-p}{p}.$$

This may be written

$$m/\mu > p + \frac{m}{\mu(\mu+1)} - \frac{1}{\mu+1} \text{ and } m/\mu < p + \frac{m}{\mu(\mu+1)}.$$

Hence as μ increases we have more and more nearly $m=\mu p$ and similarly $n=\mu q$ or $(\mu p/m)=1$ and $(\mu q/n)=1$.

Substituting these values in (3) we have

$$T_n(max)=G=\frac{1}{\sqrt{2\pi\mu pq}} \text{ and}$$

$$(6)\qquad T_n = G\left(\frac{\mu p}{m}\right)^{m+\frac{1}{2}}\left(\frac{\mu q}{n}\right)^{n+\frac{1}{2}}.$$

Let $G_{n\pm l}$ be the value which $T_{n\pm l}$ takes for $m=\mu p$; $n=\mu q$. Therefore,

$$(7)\qquad G_{n-l}=G[1+(l/m)]^{-m-l-\frac{1}{2}}[1-(l/n)]^{-n+l-\frac{1}{2}}.$$

Since we always have $|l| < m$ we have

$$[1+(l/m)]^{-m-l-\frac{1}{2}} = e^{(-m-l-\frac{1}{2})\log(1+l/m)} = e^{[(-m-l-\frac{1}{2})/m]+[(ml^2+l^3+\frac{1}{2}l^2)/(2m^2)]\cdots\cdots}$$

Substituting in (7) we obtain

$$(8)\qquad \begin{cases} G_{n-l} = G.e^{\left\{\frac{-l^2+l}{2m}+\frac{l^3+\frac{3}{2}l^2}{m^2}\cdots\cdots\frac{-l^2-l}{2n}+\frac{-l-\frac{3}{2}l^2}{6n^2}\cdots\cdots\right\}} \\ G_{n+l} = G.e^{\left\{\frac{-l^2-l}{2m}+\frac{-l^3-\frac{3}{2}l^2}{6m^2}\cdots\cdots\frac{-l^2-l}{2n}+\frac{l^3+\frac{3}{2}l^2}{6n^2}\cdots\cdots\right\}} \end{cases}$$

The number of positive errors in G_{n-l} is

$$(9)\qquad \begin{cases} \mu p + l \text{ and negative } \mu q - l. \quad \text{In } G_{n+l} \text{ the positive are} \\ \mu p - l \text{ and negative } \mu q + l. \end{cases}$$

By our hypothesis (4) $p=q=\frac{1}{2}$. Then we get from (8), neglecting terms higher than the first order in $1/\mu$.

(10) $$G_{n-l}=G_{n+l}=Ge^{-l^2/\frac{1}{2}\mu}.$$

Suppose that the absolute value of the error produced by a unit influence is α, then the errors in this case are by (9),

(11) $$\begin{cases}(\frac{1}{2}\mu+l)\alpha-(\frac{1}{2}\mu-l)\alpha=2l\alpha=x\\(\frac{1}{2}\mu-l)\alpha-(\frac{1}{2}\mu+l)\alpha=-2l\alpha=-x.\end{cases}$$

Increasing l by unity increases the right side by $2\alpha=\triangle x$, a very small quantity from the nature of the problem. From $2\alpha=\triangle x$ and (11) we have

(12) $$l=x/\triangle x.$$

Hence (10) becomes

(13) $$G_{n-l}=G_{n+l}=Ge^{-\{x^2/[(\frac{1}{2}\mu)\triangle x^2]\}}$$

We may, without loss of generality, suppose α and consequently x to become indefinitely small, and at the same time μ will become indefinitely great. The absolute value of one unit error is $\frac{1}{2}\triangle x$, and if they are all of the same character their sum, $\mu(\triangle x/2)$, which is the maximum error possible, becomes from the nature of the problem an indefinitely large quantity. Hence $(\frac{1}{2}\mu)(\triangle x)^2$ equals a finite constant, say, $1/h^2$. Then we have

(14) $$G=(h/\sqrt{\pi})dx$$

when $\triangle x$ becomes dx. This is the probability of the error zero. Substituting in (13) we have

(15) $$G_{n-l}=G_{n+l}=(h/\sqrt{\pi})e^{-h^2x^2}dx=px,$$

which is the probability of an error with the magnitude x. This is the well-known probability formula.

II. *The method of obtaining the most probable standing of each contestant.*

Let $w_1, w_2 \ldots\ldots w_n$ be the markings given to a contestant by the respective judges. Let z be the true marking which the contestant deserves.

Let $z-w_1=\varepsilon_1$; $z-w_2=\varepsilon_2$ $\ldots\ldots z-w_n=\varepsilon_n$. Then by (15) the probability that the error ε_i will occur is

$$p\varepsilon_i=\frac{h}{\sqrt{\pi}}e^{-h^2\varepsilon_i^2}d\varepsilon_i.$$

The probability that the errors $\varepsilon_1, \varepsilon_2 \ldots\ldots \varepsilon_n$ will occur together in one marking of each judge is,

(16) $$P=p\varepsilon_1.p\varepsilon_2\ldots\ldots p\varepsilon_n=\frac{h^n}{(\sqrt{\pi})^n}e^{-h^2(\varepsilon_1^2+\varepsilon_2^2+\ldots+\varepsilon_n^2)}d\varepsilon_1 d\varepsilon_2\ldots d\varepsilon_n$$

The most probable value of z is that one which will make P a maximum. P is a maximum when $(\varepsilon_1^2+\varepsilon_2^2+\ldots\ldots\varepsilon_n^2)$ is a minimum; or, by equating to zero, the first derivative, when $\varepsilon_1+\varepsilon_2+\ldots\ldots\varepsilon_n=0$. That is when $z-w_1+z-w_2+\ldots\ldots z-w_n=0$. Solving for z we find

$$z=(w_1+w_2+\ldots\ldots w_n)/n. \tag{17}$$

Hence it appears that in the long run the best way to solve the markings of judges of contests is simply to take the arithmetical means of the markings when the merits of the contestants are compared with ideal standards.

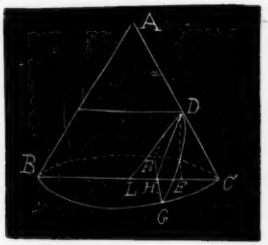
The methods for successive approximations have already been discussed in this journal.

The University of Chicago.

A METHOD OF DEFINING THE ELLIPSE, HYPERBOLA AND PARABOLA AS CONIC SECTIONS.

By W. W. LANDIS, A. M., Professor of Mathematics in Dickinson College, Carlisle, Pa.

ABC is a right circular cone, the angle at the vertex being 2α. DFG is a plane section making an angle θ with the axis of the cone. Take D as the origin, DH as the axis of x, and a perpendicular through D as the axis of y. We seek to find a relation between x and y, the parameters being $g(=AD)$ and α, and the variable parameter θ. In the circle BGC, $y^2=(a\pm b)c=ac\pm bc\ldots\ldots(1)$, where $a=BL$, $b=LH$, and $c=HC$. In the isosceles triangle DLC, $x^2=d^2\mp bc\ldots\ldots(2)$, where $d=DC=DL$. Adding (1) and (2), $x^2+y^2=ac+d^2\ldots\ldots(3)$.



Now $c=x\sin\theta+d\sin\alpha$,

$d=x\cos\theta\sec\alpha=x\cos\theta/\cos\alpha$, and

$a=2g\sin\alpha$.

Making this substitution we get

$$x^2+y^2=x^2\frac{\cos^2\theta}{\cos^2\alpha}+2g\sin\alpha\left[x\sin\theta+x\frac{\cos\theta\sin\alpha}{\cos\alpha}\right]$$

$$\text{or } x^2\left[1-\frac{\cos^2\theta}{\cos^2\alpha}\right]+y^2-2gx\sin\alpha[\sin\theta+\cos\theta\tan\alpha]=0\ldots\ldots(4)$$

which we may write

$$x^2\left[\frac{\sin^2\theta-\sin^2\alpha}{1-\sin^2\alpha}\right]+y^2-2gx\sin\alpha[\sin\theta+\cos\theta\tan\alpha]=0\ldots\ldots(5).$$

Now if $\theta=\alpha$, the plane then being parallel to one and only one element, (5) reduces to $y^2-4gx\sin^2\alpha=0$, a parabola of latus rectum $=4g\sin^2\alpha$.

If $\theta>\alpha$ the section cuts all elements and the coefficients of x^2 and y^2 are both positive, and we have an ellipse whose center, axes, and eccentricity are readily found; and in particular if $\theta=90°$, the section is parallel to the base, the coefficients of x^2 and y^2 are unity and we have a circle, whose center is $(g\sin\alpha, 0)$. If $\theta<\alpha$, the section cuts both nappes, the coefficients of x^2 and y^2 are of unlike sign and we have a hyperbola.

If $g=0$, (5) becomes $y=\pm x\sqrt{\dfrac{\sin^2\alpha-\sin^2\theta}{\cos^2\alpha}}$ ....(6). Now if $\theta=\alpha$, (6) becomes $y=0$, a straight line, the limit of the parabola. If $\theta<\alpha$, (6) represents two real straight lines, the limiting case of the hyperbola. And if $\theta>\alpha$, (6) represents two imaginary lines intersecting in the real point (0, 0), which is the limiting form of the ellipse.

The equations (5) and (6) show the dependence of the nature of the conic sections upon the angle which the cutting plane makes with the axis, and the dependence of their shape upon the angle of the cone and the distance from the vertex to the first element cut.

REMARK 1. If the section be a parabola, the foot of the perpendicular from the middle point of the line through D parallel to BC, upon DH, is the focus.

REMARK 2. The eccentricity of any conic section is $\varepsilon=[\cos\theta/\cos\alpha]$.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from November Number.]

LXX. Fig. 31.

$AEM=ACB$ of $AEHC$.

$MOL=ACB=ADK+DKBC=BHI+DKBC$. $LOI=BEK$.

$\therefore ABLM=\bigcirc BCDF+AEHC$.

LXXI. Fig. 31.

$AMPQ=\bigcirc AMOC=\bigcirc AEHC$. $BLPQ=\bigcirc BLOC=\bigcirc BCDF$.

$\therefore ABLM=\bigcirc BCDF+AEHC$.

LXXII. Fig. 31.

$MTE=BSF$. $\therefore BS=MT$. $\therefore AMLB=\bigcirc 2AMTS$.

But $AMTS=\bigcirc EAC+FBC$; since $MTE=BSF$, and $AEM=ACB$.

$\therefore 2AMTS=\bigcirc 2EAC+2FBC=\bigcirc AEHC+BCDF$.

$\therefore AMLB=\bigcirc BCDF+AEHC$.

LXXIII. Fig. 31.

$ABVU \eqsim ABHW$. $UVLM \eqsim 2MLH \eqsim 2MOL + 2LOH \eqsim HEW + ACB + 2FBC (\eqsim BCDF)$.

$\therefore ABLM \eqsim BCDF + AEHC$.

LXXIV. Fig. 31.

$ABLM \eqsim CZYX - 4ACB$.

$BCDF + AEHC = OLZH + AEHC \eqsim CZYX - 2MOLY \eqsim CZYX - 4ACB$.

$\therefore ABLM \eqsim BCDF + AEHC$.

NOTE.—This proof is similar to that of Henry Boad's, London, 1733.

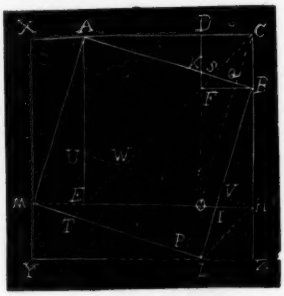


Fig. 31.

LXXV. Fig. 32.

$ANML \eqsim 2ACL = 2AFB \eqsim AFHC$.

$NMKB \eqsim 2BCK \eqsim BCDE$.

$\therefore ABKL \eqsim BCDE + AFHC$. *Wipper.*

LXXVI. Fig. 32.

Since EC produced to O passes through the center of $ABKL$, $ABQO \eqsim \frac{1}{2} ABKL$. Now, $APCO \eqsim CAF$, since $ACO \Rightarrow AFP$; $PBQC \eqsim CBE$, since $BCP = BEQ$. $\therefore ABKL \eqsim BCDE + AFHC$.

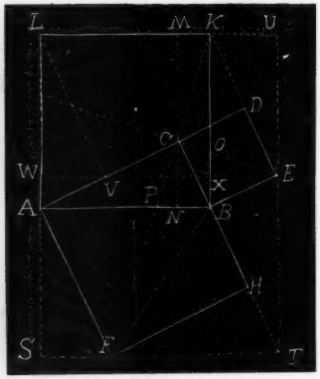


Fig. 32.

LXXVII. Fig. 32.

$LVDK = AFHB$. Then, $AVL = BEK$.

$\therefore ABKL \eqsim BCDE + AFHC$.

LXXVIII. Fig. 32.

$ABKL \eqsim STUL - 2ASF - 3FHT - AFHB$.

$BCDE + AFHC \eqsim STUL - 2ASF - 3FHT - LVDK (= AFHB)$.

$\therefore ABKL \eqsim BCDE + AFHC$.

LXXIX. Fig. 32.

$ABXW \eqsim ABEV \eqsim BCDE$.

$WXKL \eqsim VEKL \eqsim AFHC$, since $VED = ABC$, and $VDKL = FHBA$.

$\therefore ABKL \eqsim BCDE + AFHC$.

LXXX. Fig. 33.

$AOML \eqsim ANCL \eqsim 2ANC \eqsim AFHC$.

$OBKM \eqsim 2BCK \eqsim BCDE$, since SK, the altitude of BCK, $= BC$.

$\therefore ABKL \eqsim BCDE + AFHC$.

LXXXI. Fig. 33.

$AOML \eqsim 2ACL = 2AFB \eqsim AFHC$.

$OBKM \eqsim BKCN \eqsim 2BCN \eqsim BCDE$.

$\therefore ABKL \eqsim BCDE + AFHC$.

LXXXII. Fig. 33.

$QTCS = PFNE$.

$ATL + LQK + KSB = APB + CDN + NHC$.

$\therefore ABKL \eqsim BCDE + AFHC$.

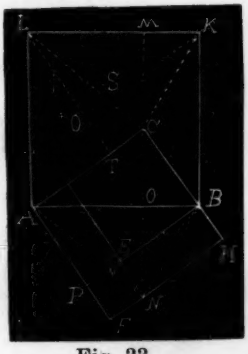


Fig. 33.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton, New Jersey.

V.

INTRODUCTION OF NEW VARIABLES IN A ONE PARAMETER GROUP. LIE'S SYMBOL OF AN INFINITESIMAL TRANSFORMATION. CANONICAL FORM AND CANONICAL VARIABLES OF A ONE PARAMETER GROUP.

1. A point transformation has been defined to be an operation by which a point (x, y) is carried to the position of some point (x_1, y_1). If this operation is applied to all the points of the xy-plane simultaneously, we have a point transformation of the plane into itself. Such an operation has been represented analytically by two equations of the form

$$x_1 = \varphi(x, y), \quad y_1 = \psi(x, y), \tag{1}$$

where x, y represent the original points and x_1, y_1 their new positions after the transformation has been effected, and where finally the fractions φ and ψ are independent analytical functions. By introducing a parameter a into the equations (1) they will no longer represent a single transformation but an infinite number of transformations forming a family of ∞^1 transformations given by

$$x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a). \tag{2}$$

If now this family contains the inverse transformation of every transformation in it and the family possesses the property that the successive performance or product of any two transformations of the family is equivalent to a single transformation belonging to the family, the equations (2) are said to define a one parameter, finite, continuous group of transformations.

The reader will observe that the transformation (1) could have been represented by the equations

$$r_1 = \rho(r, \theta), \quad \theta_1 = \sigma(r, \theta), \tag{3}$$

where r, θ are the polar coördinates of the initial position of the point and r_1, θ_1 the coördinates of its final position, the functions ρ and σ being independent analytical functions. Similarly the one parameter group of transformations (2) could have been represented in polar coördinates by the two equations

$$r_1 = \rho(r, \theta, a), \quad \theta_1 = \sigma(r, \theta, a), \tag{4}$$

where a is a parameter. Hence we see that the operation (1) and the group property of the family (3) are independent of the coördinate system to which the points of the plane are referred. Accordingly, if in the equations

$$x_1=\varphi(x, y, t), \quad y_1=\psi(x, y, t) \tag{5}$$

of a one parameter group new variables $\bar{x}$, $\bar{y}$, $\bar{x}_1$, $\bar{y}_1$ are introduced, where

$$\left.\begin{aligned} \bar{x}&=\lambda(x, y), \quad \bar{y}=\mu(x, y), \\ \bar{x}_1&=\lambda(x_1, y_1), \quad \bar{y}_1=\mu(x_1, y_1), \end{aligned}\right\} \tag{6}$$

the new equations

$$x_1=\Phi(x, y, t), \quad y_1=\Psi(x, y, t) \tag{7}$$

obtained from the original ones (3) by these substitutions (6) will also represent a one parameter group.

Thus for example, it is easy to show that the ∞^1 transformations given by the equations

$$x_1=x+t, \quad y_1=\frac{xy}{x+t} \tag{8}$$

represent a one parameter group, for if (x_1, y_1) be transformed into (x_2, y_2) we have

$$x_2=x_1+t_1, \quad y_2=\frac{x_1y_1}{x_1+t_1}. \tag{9}$$

Eliminating the variables (x_1, y_1) from the equations (9) by means of the equations (8) we find

$$x_2=x+(t+t_1), \quad y_2=\frac{xy}{x+(t+t_1)}.$$

These last equations prove that the family (8) is a group, since they represent the product of two transformations of the family and are of the same form as the defining equations (8) of the family.

Now let us introduce new variables x, y into the equations of the group defined by the equations

$$\bar{x}=x/y, \quad \bar{y}=xy; \tag{10}$$

then putting

$$\bar{x}_1=x_1/y_1, \quad \bar{y}_1=x_1y_1, \tag{11}$$

and eliminating x, y, x_1, y_1 from the equations (8), (10) and (11), we have

$$\left.\begin{aligned} \bar{x}_1&=\frac{(x+t)^2}{xy}=\frac{(\sqrt{\bar{x}\bar{y}}+t)^2}{\bar{y}}, \\ \bar{y}_1&=xy=\bar{y}. \end{aligned}\right\} \tag{12}$$

The point $(\bar{x}, \bar{y})$ is carried to the position $(\bar{x}_1, \bar{y}_1)$ by the transformation (12) of the family (12), and a transformation corresponding to the parameter t_1 carries $(\bar{x}_1, \bar{y}_1)$ to the position $(\bar{x}_2, \bar{y}_2)$, say, where

$$\left.\begin{aligned} \bar{y}_2 &= \frac{(\sqrt{\bar{x}_1}, \bar{y}_1 + t_1)^2}{\bar{y}_1}, \\ \bar{y}_2 &= \bar{y}_1. \end{aligned}\right\} \qquad (13)$$

The elimination of $\bar{x}_1$, $\bar{y}_1$ by means of the equations (12) and (13) gives,

$$\left.\begin{aligned} \bar{x}_2 &= \frac{(\sqrt{\bar{x}\bar{y}} + t + t_1)^2}{y}, \\ \bar{y}_2 &= \bar{y}, \end{aligned}\right\} \qquad (14)$$

the equations of the transformation of the family (12) which carries the point $(\bar{x}, \bar{y})$ directly to the position $(\bar{x}_2, \bar{y}_2)$; the form of these equations shows that this transformation belongs to the family; hence the family is a group.

2. The reader will recall that we have established in preceding sections LIE's theorem relative to the connection between the notions infinitesimal transformation and one parameter group. This relation proved to be a one-to-one correspondence of such an intimate relation that the infinitesimal transformation of a one parameter group may be taken as the complete representative of the latter without loss of generality or property. In order to make the fullest use of this equivalence of notions LIE has devised a very convenient symbol for an infinitesimal transformation. This symbol is readily constructed as follows:

Let the finite equations of the group be

$$x_1 = \varphi(x, y, t), \quad y_1 = \psi(x, y, t); \qquad (15)$$

and those of its infinitesimal transformation be

$$x' = x_1 + \xi(x_1, y_1)\delta t + \ldots\ldots, \quad y' = y_1 + \eta(x_1, y_1)\delta t \ \ldots\ldots; \qquad (16)$$

and let the identical transformation of the group correspond to the zero value of the parameter t.

By virtue of the infinitesimal transformation (16), x_1 and y_1 receive the increments

$$\delta x_1 = \xi(x_1, y_1)\delta t, \quad \delta y_1 = \eta(x_1, y_1)\delta t. \qquad (17)$$

The increment which an arbitrary function $f(x_1, y_1)$ receives by this infinitesimal transformation is found by substituting these values (17) in

$$\delta f_1 = \frac{\partial f(x_1, y_1)}{\partial x_1}\delta x_1 + \frac{\partial f(x_1, y_1)}{\partial y_1}\delta y_1,$$

to be

$$\delta f_1 = \{\xi(x_1, y_1)\frac{\partial f(x_1, y_1)}{\partial x_1} + \eta(x_1, y_1)\frac{\partial f(x_1, y_1)}{\partial y_1}\}\delta t;$$

since $f(x_1, y_1)$ becomes $f(x, y)$ when $t=0$, we have for the increment of the function $f(x, y)$

$$\delta f=\{\xi(x, y)\frac{\partial f}{\partial x}+\eta(x, y)\frac{\partial f}{\partial y}\}\delta t. \tag{18}$$

Conversely, when the increment which an arbitrary function receives by an infinitesimal transformation is known, the infinitesimal transformation itself is known ; for putting $f\equiv x$, (18) becomes

$$\delta t=\xi(x, y)\delta t\,;$$

and putting $f\equiv y$,

$$\delta t=\eta(x, y)\delta t.$$

Hence instead of defining an infinitesimal transformation by means of two equations (16) it may be characterized by the expression

$$\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y},$$

which is equal to the quotient of the increment of an arbitrary function $f(x, y)$ by the infinitesimal transformation divided by δt. For this reason LIE adopts the symbol

$$Uf\equiv\frac{\delta f}{\delta t}=\xi(x, y)\frac{\partial f}{\partial x}+\eta(x, y)\frac{\partial f}{\partial y} \tag{19}$$

as the symbol of the infinitesimal transformation (16).

If we replace $f(x, y)$ by x in this expression we find $Ux=\xi$, and if we put y for f we have $Uy=\eta$, hence the symbol may be written

$$Uf=Ux\frac{\partial t}{\partial x}+Uy\frac{\partial t}{\partial y}. \tag{20}$$

3. The question arises—How is the form of this symbol (19) affected by the introduction of new variables ? The solution of this question leads to one of the remarkable properties of the symbol.

Let the new variables be

$$\bar{x}=\lambda(x, y), \quad \bar{y}=\mu(\bar{x}, \bar{y})\,;$$

by their substitution the function $f(x, y)$ becomes $f(\bar{x}, \bar{y})$, and the principles of partial differentiation give

$$\frac{\partial t}{\partial x}=\frac{\partial f}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x}+\frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial x}, \qquad \frac{\partial f}{\partial y}=\frac{\partial f}{\partial \bar{x}}\frac{\partial x}{\partial y}+\frac{\partial f}{\partial \bar{y}}\frac{\partial y}{\partial y}.$$

Hence

$$Uf \equiv \xi(x,\ y)\frac{\partial f}{\partial x} + \eta(x,\ y)\frac{\partial f}{\partial y}$$

becomes

$$\xi\left(\frac{\partial t}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} + \frac{\partial t}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial x}\right) + \eta\left(\frac{\partial f}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial y} + \frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial y}\right);$$

or

$$\left(\xi\frac{\partial \bar{x}}{\partial x} + \eta\frac{\partial \bar{x}}{\partial y}\right)\frac{\partial f}{\partial \bar{x}} + \left(\xi\frac{\partial \bar{y}}{\partial x} + \eta\frac{\partial \bar{y}}{\partial y}\right)\frac{\partial f}{\partial \bar{y}};$$

or finally designating Uf in the new variables by $\bar{U}f$,

$$\bar{U}f = U\bar{x}\frac{\partial f}{\partial \bar{x}} + U\bar{y}\frac{\partial f}{\partial \bar{y}}.$$

Hence if new variables, $\bar{x}$, $\bar{y}$, $\bar{x}_1$, $\bar{y}_1$ are introduced into the equations of a one parameter group,

$$x_1 = \varphi(x,\ y,\ t), \quad y_1 = \psi(x,\ y,\ t),$$

by means of the equations,

$$\bar{x} = \lambda(x,\ y), \quad \bar{y} = \mu(x,\ y),$$

$$\bar{x}_1 = \lambda(x,\ y), \quad \bar{y}_1 = \mu(x,\ y),$$

the symbol $\bar{U}f$ of the infinitesimal transformation of the new group can be determined directly by introducing the variables $\bar{x}$, $\bar{y}$ into the symbol Uf of the original group.

4. The result of the preceding paragraph gives rise to the fact that by introducing new variables $\bar{x}$, $\bar{y}$ any given infinitesimal transformation

$$Uf \equiv \xi(x,\ y)\frac{\partial f}{\partial x} + \eta(x,\ y)\frac{\partial f}{\partial y}$$

can be brought to the form of any other

$$\bar{U}f \equiv \bar{\xi}(\bar{x},\ \bar{y})\frac{\partial f}{\partial \bar{x}} + \bar{\eta}(\bar{x},\ \bar{y})\frac{\partial f}{\partial \bar{y}}.$$

In order to make this transformation it is only necessary to choose $\bar{x}$ and $\bar{y}$ as such functions of x, y that the identity

$$\bar{\xi}\frac{\partial f}{\partial \bar{x}} + \bar{\eta}\frac{\partial f}{\partial \bar{y}} = U\bar{x}\frac{\partial f}{\partial \bar{x}} + U\bar{y}\frac{\partial f}{\partial \bar{y}},$$

shall exist for all values of f. Applying the theorem of undetermined coefficients the preceding identity must break up into the two

$$U\bar{x}=\bar{\xi}, \quad U\bar{y}=\bar{\eta},$$

or written in full,

$$\xi\frac{\partial\bar{x}}{\partial x}+\eta\frac{\partial\bar{x}}{\partial y}=\bar{\xi}, \qquad \xi\frac{\partial\bar{y}}{\partial x}+\eta\frac{\partial\bar{y}}{dy}=\bar{\eta}.$$

These are two partial differential equations and we know that they have independent solutions if $\bar{\xi}$ and $\bar{\eta}$ are not both identically zero ; hence the proposed transformation is always possible since Uf is the representative of the original group and $\bar{U}f$ that of the transformed group. This proves LIE's theorem :

By introducing new variables every one parameter group of the plane can be changed into every other one parameter group of the same plane.

5. If $\bar{U}f$ has the particular form

$$\bar{U}f\equiv\frac{\partial f}{\partial\bar{y}},$$

we have the remarkable theorem that *every one parameter group of the plane can be brought to the form of a group of translations by a proper change of variables.*

These new variables $\bar{x}$ and $\bar{y}$ are found by integrating the differential equations

$$U\bar{x}\equiv\xi\frac{\partial\bar{x}}{\partial x}+\eta\frac{\partial\bar{x}}{\partial y}=0, \quad U\bar{y}\equiv\xi\frac{\partial\bar{y}}{\partial x}+\eta\frac{\partial\bar{y}}{\partial y}=1, \qquad (21)$$

since in the form $\bar{U}f\equiv\frac{\partial f}{\partial\bar{y}}$, $\bar{\xi}=0$, and $\bar{\eta}=1$.

Now if we recall the fact that a partial differential equation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0$$

is equivalent to the ordinary differential equation

$$\frac{dx}{x}=\frac{dy}{y},$$

and that the partial differential equation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=1$$

is equivalent to the simultaneous system

$$\frac{dx}{x}=\frac{dy}{y}=df,$$

the equations (21) can be replaced by the following

$$\frac{dx}{\xi}=\frac{dy}{\eta}, \tag{22}$$

$$\frac{dx}{\xi}=\frac{dy}{\eta}=d\bar{y}. \tag{23}$$

The integration of (22) gives the function $\bar{x}$ say. This substituted in (23), the latter can be integrated by a quadrature and determines $\bar{y}$.

That the infinitesimal transformation

$$\bar{U}f\equiv\frac{\partial f}{\partial \bar{y}}$$

generates a group of translations parallel to the $\bar{y}$-axis is easily shown by applying a theorem of LIE proved in a previous paragraph, namely, the theorem which shows how to construct the finite equations of a group when those of its infinitesimal transformation are known. The simultaneous system to be integrated in this case has the form

$$\frac{d\bar{x}_1}{0}=\frac{d\bar{y}_1}{1}=dt,$$

with the initial values $\bar{x}$, $\bar{y}$, 0. The integration of this simple system gives the finite equation in the form

$$\bar{x}_1=\bar{x}, \quad \bar{y}_1=\bar{y}+t;$$

that is, the group is a group of translations parallel to the $\bar{y}$-axis.

6. The theorem of the preceding paragraph can be proved by using the finite equations of the group. We have found in a previous article of this series that the finite equations

$$x_1=\varphi(x, y, t), \quad y_1=\psi(x, y, t) \tag{24}$$

of a one parameter group are found by integrating a certain simultaneous system in the form*

$$\Omega(x_1, y_1)=\Omega(x, y),$$

$$W(x_1, y_1)+t=W(x, y).$$

*The reader will observe that this is the same theorem made use of in the paragraph immediately preceding.

Now if the functions Ω and W be introduced as new variables in place of x and y, and put

$$\bar{x}=\Omega(x, y), \quad \bar{y}=W(x, y),$$

$$\bar{x}_1=\Omega(x_1, y_1), \quad \bar{y}_1=W(x_1, y_1),$$

the group takes the simple form

$$\bar{x}_1=\bar{x}, \quad \bar{y}_1=\bar{y}+t; \tag{25}$$

that is, the form of a group of translations.

The new variables $\bar{x}$ and $\bar{y}$ are called the *canonical variables* of the one parameter group (24), and (25) is called the *canonical form* of (24).

The reader will find many interesting details relative to the points and theorems here discussed, in the third chapter of LIE's lectures on differential equations.

Princeton University, 21 February, 1898.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

84. Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.

Show how to find sides, integral, fractional, and irrational for twenty-four triangles, each one containing 330 square yards.

Solution by the PROPOSER.

1. $\triangle^2=330^2=2\times2\times3\times3\times5\times5\times11\times11=(5\times11)(2\times3\times5)(2\times11)\times3$ $=55\times30.22\times3$. $\therefore$ Sides of triangle are 25, 33, and 52.

2. $\triangle^2=330^2=(2\times5\times11)(3\times3\times11)(2\times5)=110\times99\times10\times1$. $\therefore$ Sides of $\triangle$ are 11, 100, and 109.

3. $\triangle^2=330^2=(2\times5\times5)(3\times11)(11)(2\times3)=50\times33\times11\times6$. $\therefore$ Sides of $\triangle$ are 17, 39, 44.

4. $330^2=(2\times3\times11)(5\times11)(2\times3)(5)=66\times55\times6\times5$. $\therefore$ Sides of $\triangle$ are 11, 60, and 61.

5. $330^2=(2\times2\times2\times2\times3)\left(\frac{5\times11}{2}\right)(3\times5)(\frac{1}{2})=48\times27\frac{1}{2}\times15\times5\frac{1}{2}$. $\therefore$ Sides of $\triangle$ are $20\frac{1}{2}$, 33, and $42\frac{1}{2}$.

6. $330^2=(2\times2\times11)(2\times2\times5)\left(\frac{3\times11}{2}\right)\left(\frac{3\times5}{2}\right)=44\times20\times16\frac{1}{2}\times7\frac{1}{2}$. $\therefore$ Sides of $\triangle$ are 24, $27\frac{1}{2}$, and $36\frac{1}{2}$.

7. $330^2=(2\times2\times3\times5)\left(\frac{3\times5\times11}{2\times2}\right)(2\times2\times2\times2)\left(\frac{11}{2\times2}\right)=60\times41\frac{1}{4}\times16\times2\frac{3}{4}$. $\therefore$ Sides of $\triangle$ are $18\frac{3}{4}$, 44, and $57\frac{1}{4}$.

8. $330^2=(3\times3\times11)(2\times3\times3\times5)\left(\frac{2\times11}{3}\right)(\frac{5}{3})=99\times90\times7\frac{1}{3}\times1\frac{2}{3}$. $\therefore$ Sides of $\triangle$ are 9, $91\frac{2}{3}$, and $97\frac{1}{3}$.

9. $\triangle^2=165^2=3\times3\times5\times5\times11\times11=(3\times11)(2\times2\times5)\left(\frac{3\times5}{2}\right)(\frac{1}{2})=33\times20\times7\frac{1}{2}\times5\frac{1}{2}$. $\therefore$ Sides of $\triangle=165$ are 13, $25\frac{1}{2}$, and $27\frac{1}{2}$, and sides of $\triangle=330$ are $13_1/2$, $25\frac{1}{2}_1/2$, and $27\frac{1}{2}_1/2$.

10. $165^2=(2\times3\times5)\left(\frac{5\times11}{2\times2}\right)\left(\frac{3\times11}{2\times2}\right)(2\times2\times2)=30\times13\frac{3}{4}\times8\frac{1}{4}\times8$. $\therefore$ Sides of $\triangle$ are $16\frac{1}{4}$, $21\frac{3}{4}$, and 22. Sides of similar $\triangle$ containing 330 must be $16\frac{1}{4}_1/2$, $21\frac{3}{4}_1/2$, and $22_1/2$.

11. $\triangle^2=110^2=2\times2\times5\times5\times11\times11=(2\times2\times2\times3)\left(\frac{5\times11}{2\times3}\right)\left(\frac{3\times5}{2}\right)\left(\frac{2\times11}{3}\right)$ $=24\times9\frac{1}{6}\times7\frac{1}{2}\times7\frac{1}{3}$. $\therefore$ Sides of $\triangle=110$ are $14\frac{5}{6}$, $16\frac{1}{2}$, and $16\frac{2}{3}$. Sides of $\triangle=330$ are $14\frac{5}{6}_1/3$, $16\frac{1}{2}_1/3$, and $16\frac{2}{3}_1/3$.

12. $110^2=\left(\frac{2\times5\times11}{3}\right)(2\times3\times5)(\frac{1}{3})(3)=36\frac{2}{3}\times30\times3\frac{2}{3}\times3$. $\therefore$ Sides are $6\frac{2}{3}$, 33, and $33\frac{2}{3}$, and sides of similar $\triangle=330$ must be $6\frac{2}{3}_1/3$, $33_1/3$, and $33\frac{2}{3}_1/3$.

13. $\triangle^2=66^2=2\times2\times3\times3\times11\times11=(2\times11)(11)(3\times3)(2)=22\times11\times9\times2$. $\therefore$ Sides of $\triangle=66$ are 11, 13, and 20, and sides of $\triangle=330$ are $11_1/5$, $13_1/5$, and $20_1/5$.

14. $66^2=(2\times3\times3\times3)(3\times3\times3)\left(\frac{2\times11\times11}{3\times3}\right)\left(\frac{1}{3\times3}\right)=54\times27\times26\frac{8}{9}\times\frac{1}{9}$. $\therefore$ Sides are 27, $27\frac{1}{9}$, and $53\frac{8}{9}$, and sides of $\triangle=330$ are $27_1/5$, $27\frac{1}{9}_1/5$, and $53\frac{8}{9}_1/5$.

15. $66^2=(3\times11)\left(\frac{3\times11}{2}\right)(2\times2\times2\times2)(\frac{1}{2})=33\times16\frac{1}{2}\times16\times\frac{1}{2}$. $\therefore$ Sides of $\triangle=66$ are $16\frac{1}{2}$, 17, and $32\frac{1}{2}$. Necessarily, sides of similar $\triangle=330$ are $16\frac{1}{2}_1/5$, $17_1/5$, and $32\frac{1}{2}_1/5$.

16. $66^2=(2\times2\times2\times3)\left(\frac{3\times11}{2}\right)(\frac{1}{2})(2)=24\times16\frac{1}{2}\times5\frac{1}{2}\times2$. $\therefore$ Sides are $7\frac{1}{2}$, $18\frac{1}{2}$, and 22, and sides of similar $\triangle=330$ are $7\frac{1}{2}_1/5$, $18\frac{1}{2}_1/5$, $22_1/5$.

17. $\triangle^2=55^2=(5\times5\times11\times11)=(2\times3\times3)\left(\frac{5\times11}{2\times3}\right)(\frac{1}{2})\left(\frac{2\times5}{3}\right)=18\times9\frac{1}{6}\times5\frac{1}{2}\times3\frac{1}{2}$. $\therefore$ Sides are $8\frac{5}{6}$, $12\frac{1}{2}$, and $14\frac{3}{4}$, and sides of $\triangle=330$ are $8\frac{5}{6}_1/6$, $12\frac{1}{2}_1/6$, and $14\frac{3}{4}_1/6$.

18. $55^2=(2\times2\times5)\left(\frac{5\times11}{2\times3}\right)(3\times3)\left(\frac{11}{2\times3}\right)=20\times9\frac{1}{6}\times9\times1\frac{5}{6}$. $\therefore$ Sides of $\triangle=55$ are $10\frac{5}{6}$, 11, and $18\frac{1}{2}$, and sides of $\triangle=330$ are $10\frac{5}{6}_1/6$, $11_1/6$, $18\frac{1}{2}_1/6$.

19. $\triangle^2=(47\frac{1}{7})^2=(\frac{330}{7})^2=(3\times11)(2\times3\times5)(\frac{1}{7})\left(\frac{2\times5}{7}\right)=33\times30\times1\frac{4}{7}\times1\frac{3}{7}$.

$\therefore$ Sides are 3, $31\frac{3}{7}$, and $31\frac{4}{7}$, and sides of $\triangle=330$ are $3\sqrt{7}$, $31\frac{3}{7}\sqrt{7}$, and $31\frac{4}{7}\sqrt{7}$.

20. $\triangle^2=(47\frac{1}{7})^2=(\frac{330}{7})=\left(\frac{2\times5\times11}{7}\right)\left(\frac{3\times11}{7}\right)(2\times3)(5)=15\frac{5}{7}\times4\frac{5}{7}\times6\times5$.

$\therefore$ Sides are 11, $9\frac{5}{7}$, and $10\frac{5}{7}$, and sides of $\triangle=330$ are $11\sqrt{7}$, $9\frac{5}{7}\sqrt{7}$, and $10\frac{5}{7}\sqrt{7}$.

21. $(47\frac{1}{7})^2=(\frac{330}{7})^2=(2\times11)\left(\frac{2\times2\times2\times3\times5}{7}\right)(\frac{5}{2})\left(\frac{3\times11}{2\times7}\right)=22\times17\frac{1}{4}\times2\frac{1}{2}$ $\times2\frac{5}{14}$. $\therefore$ Sides are $4\frac{6}{7}$, $19\frac{1}{2}$, and $19\frac{9}{14}$, and sides of $\triangle=330$ are $4\frac{6}{7}\sqrt{7}$, $19\frac{1}{2}\sqrt{7}$, $19\frac{9}{14}\sqrt{7}$.

22. $(47\frac{1}{7})^2=(\frac{330}{7})^2=\left(\frac{3\times11}{2}\right)\left(\frac{2\times2\times3\times5}{7}\right)(2\times2)\left(\frac{5\times11}{2\times7}\right)=16\frac{1}{2}\times8\frac{4}{7}\times4$ $\times3\frac{13}{14}$. $\therefore$ Sides are $7\frac{13}{14}$, $12\frac{1}{2}$, and $12\frac{4}{7}$, and sides of $\triangle=330$ are $7\frac{13}{14}\sqrt{7}$, $12\frac{1}{2}\sqrt{7}$, and $12\frac{4}{7}\sqrt{7}$.

23. $\triangle^2=33^2=(3\times3\times11\times11)=\left(\frac{3\times11}{2}\right)(11)(2\times2)(\frac{3}{2})=16\frac{1}{2}\times11\times4\times1\frac{1}{2}$.

$\therefore$ Sides are $5\frac{1}{2}$, $12\frac{1}{2}$, and 15, and sides of similar $\triangle=330$ are $5\frac{1}{2}\sqrt{10}$, $12\frac{1}{2}\sqrt{10}$, and $15\sqrt{10}$.

24. $33^2=\left(\frac{3\times3\times5}{2}\right)\left(\frac{11\times11}{2\times5}\right)(2\times5)(\frac{2}{5})=22\frac{1}{2}\times12\frac{1}{10}\times10\times\frac{2}{5}$. $\therefore$ Sides are $10\frac{2}{5}$, $12\frac{1}{2}$, and $22\frac{1}{10}$, and sides of $\triangle=330$ are $10\frac{2}{5}\sqrt{10}$, $12\frac{1}{2}\sqrt{10}$, and $22\frac{1}{10}\sqrt{10}$.

25. $\triangle^2=30^2=2\times2\times3\times3\times5\times5=(3\times5)(2\times5)(3)(2)=15\times10\times3\times2$.

$\therefore$ Sides of $\triangle=30$ are 5, 12, and 13, and sides of $\triangle=330$ are $5\sqrt{11}$, $12\sqrt{11}$, and $13\sqrt{11}$.

26. $30^2=(2\times2\times5)(2\times2\times3)\left(\frac{3\times5}{2}\right)(\frac{1}{2})=20\times12\times7\frac{1}{2}\times\frac{1}{2}$. $\therefore$ Sides are 8, $12\frac{1}{2}$, and $19\frac{1}{2}$, and sides of $\triangle=330$ are $8\sqrt{11}$, $12\frac{1}{2}\sqrt{11}$, and $19\frac{1}{2}\sqrt{11}$.

27. $30^2=(2\times2\times2\times2\times2)(2\times3\times5)\left(\frac{5}{2\times2}\right)\left(\frac{3}{2\times2}\right)=32\times30\times1\frac{1}{4}\times\frac{3}{4}$.

$\therefore$ Sides of $\triangle=30$ are 2, $30\frac{3}{4}$, and $31\frac{1}{4}$, and sides of $\triangle=330$ are $2\sqrt{11}$, $30\frac{3}{4}\sqrt{11}$, and $31\frac{1}{4}\sqrt{11}$.

28. $\triangle^2=(25\frac{5}{13})^2=(\frac{330}{13})^2=(3\times5)(11)\left(\frac{2\times3\times5}{13}\right)\left(\frac{2\times11}{13}\right)=15\times11\times2\frac{4}{13}\times$ $1\frac{9}{13}$. $\therefore$ Sides are 4, $12\frac{9}{13}$, and $13\frac{4}{13}$, and sides of $\triangle=330$ are $4\sqrt{13}$, $12\frac{9}{13}\sqrt{13}$, and $13\frac{4}{13}\sqrt{13}$.

29. $(\frac{330}{13})^2=\left(\frac{2\times3\times5\times11}{13}\right)(2\times11)(3)(\frac{5}{13})=25\frac{5}{13}\times22\times3\times\frac{5}{13}$. $\therefore$ Sides are $3\frac{5}{13}$, $22\frac{5}{13}$, and 25, and sides of $\triangle=330$ are $3\frac{5}{13}\sqrt{13}$, $22\frac{5}{13}\sqrt{13}$, and $25\sqrt{13}$.

30. $\triangle^2=(23\frac{4}{7})^2=(\frac{165}{7})^2=\frac{3\times3\times5\times5\times11\times11}{7\times7}=\left(\frac{3\times5\times11}{2\times7}\right)(\frac{1}{2})$ $\left(\frac{2\times2\times3\times5}{2\times7}\right)(2)=11\frac{1}{14}\times5\frac{1}{2}\times4\frac{4}{14}\times2$. $\therefore$ Sides of $\triangle=23\frac{4}{7}$ are $6\frac{2}{7}$, $7\frac{1}{2}$, and $9\frac{1}{14}$, and sides of $\triangle=330$ are $6\frac{2}{7}\sqrt{14}$, $7\frac{1}{2}\sqrt{14}$, and $9\frac{1}{14}\sqrt{14}$.

31. $(23\frac{4}{7})^2=(1\frac{6}{7}5)^2=(2\times11)\left(\frac{3\times5\times11}{2\times7}\right)(2\times5)\left(\frac{3}{2\times7}\right)=22\times1\frac{1}{14}\times10\times\frac{3}{14}$.

$\therefore$ Sides are $10\frac{3}{14}$, 12. and $21\frac{1}{14}$, and sides of $\triangle=330$ are $10\frac{3}{14}\sqrt{14}$, $12\sqrt{14}$, and $21\frac{1}{14}\sqrt{14}$.

32. $\triangle^2=22^2=2\times2\times11\times11=(2\times11)(2\times3\times3)(\frac{1}{3}\frac{1}{3})(\frac{1}{3})=22\times18\times3\frac{2}{3}\times\frac{1}{3}$. $\therefore$ Sides are 4, $18\frac{2}{3}$. and $21\frac{2}{3}$, and sides of $\triangle=330$ are $4\sqrt{15}$, $18\frac{2}{3}\sqrt{15}$, and $21\frac{2}{3}\sqrt{15}$.

33. $\triangle^2=15^2=3\times3\times5\times5=(2\times5)(2\times3)(\frac{5}{2})(\frac{3}{2})=10\times6\times2\frac{1}{2}\times1\frac{1}{2}$. $\therefore$ Sides are 4, $7\frac{1}{2}$, and $8\frac{1}{2}$, and, of course, sides of $\triangle=330$ are $4\sqrt{22}$, $7\frac{1}{2}\sqrt{22}$, and $8\frac{1}{2}\sqrt{22}$.

34. $\triangle^2=10^2=2\times2\times5\times5=(3\times3)\ (5)\left(\frac{2\times5}{3}\right)(\frac{2}{3})=9\times5\times3\frac{1}{3}\times\frac{2}{3}$. $\therefore$ Sides are 4, $5\frac{2}{3}$, and $8\frac{1}{3}$, and sides of $\triangle=330$ are $4\sqrt{33}$, $5\frac{2}{3}\sqrt{33}$, and $8\frac{1}{3}\sqrt{33}$.

ALGEBRA.

79. Proposed by C. W. M. BLACK, A. M., Professor of Mathematics, Wesleyan Academy, Wilbraham, Mass.

Of n persons A, B, C, etc., A gives to the others as much as each of them already has; then B gives to the others as much as each then has; and so on for each in turn. Finally, A, B, C, etc., have respectively a, b, c, etc., dollars. How much had each at first?

I. Solution by the PROPOSER.

Let $a+b+c+\text{etc.}\ldots\ldots=s$, which is constant. Consider the rth person, K, who has at the end k dollars. Let x=number of dollars he had at first. Before he gives away any, his original amount is doubled $r-1$ times and comes to equal $2^{r-1}x$. At that time all the rest must together have $s-2^{r-1}x$, and if he gives away that amount he has left $2^{r-1}x-(s-2^{r-1}x)=2^rx-s$. Afterwards his money is doubled $n-r$ times and finally equals $2^{n-r}(2^rx-s)=k$.

$\therefore\ x=(k+2^{n-r}s)/2^n$.

II. Solution by I. H. BRYANT, A. M., Instructor in Mathematics in Fort Smith High School, Fort Smith, Ark.; G. B. M. ZERR, A. M., Ph. D., Russell College, Lebanon, Va.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let x_1, x_2, x_3, x_4, $\ldots\ldots$, x_n be the respective amounts each had at first.

Let $s=x_1+x_2+x_3+\ldots\ldots+x_n=a+b+c+\ldots\ldots(1)$.

$\therefore\ 2^{n-1}(2x_1-s)=a\ \ldots\ldots(2)$.

$2^{n-2}(4x_2-s)=b\ \ldots\ldots(3)$.

$2^{n-3}(8x_3-s)=c\ \ldots\ldots(4)$.

$\ldots\ldots\ \ldots\ldots\ldots\ldots$

$(2^nx_n-s=n$.

(2), (3), (4), (5), etc., in (1) gives, $x_1=(a+2^{n-1}s)/2^n$, $x_2=(b+2^{n-2}s)/2^n$, $x_3=(c+2^{n-3}s)/2^n$, $x_4=(d+2^{n-4}s)/2^n$, $\ldots\ldots x_n=(n+s)/2^n$.

III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let x_1, x_2, x_3, $\ldots\ldots$ x_{n-1}, x_n=the money that A, B, C, etc., respectively, had at first. Put $x_n=r$, $x_{n-1}=2r-s$, $x_{n-2}=2x_{n-1}-s=4r-3s$, $x_{n-3}=2x_{n-2}-s=8r-7s$, etc., etc. The law that governs the forming of these values makes $x_1=2^{n-1}(r-s)+s$; $x_2=2^{n-2}(r-s)+s$; $x_3=2^{n-3}(r-s)+s$; $x_4=2^{n-4}(r-s)+s$;

etc., etc. Whence we deduce, $a=2^{n-1}[r-(n-1)s]$; $b=2^{n-2}\{r-[n-(2^2-1)]s\}$; $c=2^{n-3}\{r-[n-(2^3-1)]s\}$; $d=2^{n-4}\{r-[n-(2^4-1)]s\}$; etc., etc. But in order that all values are positive, we have the condition $r>(n-1)s$. Therefore, put $r=(n-1)s+t$, and we obtain,

$a=2^{n-1}t$, $\qquad x_1=2^{n-1}[s(n-2)+t]+s$,

$b=2^{n-2}(2s+t)$, $\qquad x_2=2^{n-2}[s(n-2)+t]+s$,

$c=2^{n-3}2(2^2-1)s+t$, $\qquad x_3=2^{n-3}[s(n-2)+t]+s$,

$d=2^{n-4}2(2^3-1)s+t$, etc. $\qquad x_4=2^{n-4}[s(n-2)+t]+s$, etc.

s and t may be any values, integral or fractional; but s may also be *zero*. When $s=0$, the money each had at first equals, respectively, what he had at the end of the transaction; or $a=x_1=2^{n-1}t$; $b=x_2=2^{n-1}t$; $\ldots\ldots x_{n-1}=2^2t$; $x_n=2t$. When $t=2s$, all have an equal sum at the end of the transaction; or $2^ns=a=b=c$, etc. For obtaining integral values this form of solution is preferable.

IV. Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, N. J.

This problem is concerned with a double n-fold series. Let A, B, C, etc., represent the amounts which each person has to start with; then after the several exchanges, we have: A's nth term $=2^{n-1}[2A-\Sigma(A)]=a$; where $\Sigma(A)$ means $(A+B+C+\ldots$ .to n letters); B's nth term equals $2^{n-2}[4B-\Sigma(A)]=b$; C's nth term $=2^{n-3}[8C-\Sigma(A)]=c$; and so on for R's nth term $=2^{n-r}[2^rR-\Sigma(A)]=\rho$; and N's nth term $=2^{n-n}[2^nN-\Sigma(A)]=\nu$.

Expanding, we may write,

$$\left.\begin{array}{l}
(2^n-2^{n-1})A-2^{n-1}B-2^{n-1}C\ldots\ldots-2^{n-1}R\ldots\ldots\ -2^{n-1}N=a\\
-2^{n-2}A+(2^n-2^{n-2})B-2^{n-2}C\ldots\ldots-2^{n-2}R\ \ldots\ldots-2^{n-2}N=b\\
-2^{n-3}A-2^{n-3}B+(2^n-2^{n-3})C\ldots\ldots-2^{n-3}R\ldots\ldots-2^{n-3}N=c\\
\ldots\ldots\ \ldots\ldots\ \ldots\ldots\\
-2^{n-r}A-2^{n-r}B-2^{n-r}C\ldots\ldots+(2^n-2^{n-r})R\ \ldots\ldots-2^{n-r}N=\rho\\
\ldots\ldots\ \ldots\ldots\\
-2^{n-n}A-2^{n-n}B-2^{n-n}C\ldots\ldots-2^{n-n}R\ldots\ldots+(2^n-2^{n-n})N=\nu
\end{array}\right\}\text{I.}$$

Whence by determinants,

$$\Delta.A=\begin{vmatrix}
a, & -2^{n-1}, & -2^{n-1}, & \ldots\ldots-2^{n-1}, & \ldots\ldots-2^{n-1}\\
b, & 2^n-2^{n-2}, & -2^{n-2}, & \ldots\ldots-2^{n-2}, & \ldots\ldots-2^{n-2}\\
c, & -2^{n-3}, & 2^n-2^{n-3}, & \ldots\ldots-2^{n-3}, & \ldots\ldots-2^{n-3}\\
\ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots\\
\rho, & -2^{n-r}, & -2^{n-r}, & \ldots\ldots2^n-2^{n-r}, & \ldots\ldots-2^{n-r}\\
\ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots\\
\nu, & -1, & -1, & \ldots\ldots-1, & \ldots\ldots2^n-1
\end{vmatrix}$$

Where Δ is the determinant of the coefficients.

Also solved by *H. C. WILKES*.

80. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Solve $1+x^4=a(1+x)^4$.

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; P. S. BERG, A. M., Principal of Schools, Larimore, N. D.; J. OWEN MAHONEY, B. E., M. Sc., Graduate Fellow in Mathematics in Vanderbilt University, Lynnville, Tenn.; CHAS. A. JONES, Torrance, Miss.; and the PROPOSER.

$1+x^4=a+4ax+6ax^2+4ax^3+ax^4$.

$x^2+(1/x^2)=(a/x^2)+(4a/x)+6a+4ax+ax^2$.

$[x+(1/x)]^2=a[x+(1/x)]^2+4a[x+(1/x)]+4a+2$.....(1).

Let $x+(1/x)=y$.....(2); $\therefore x^2-xy=1$.

$\therefore x=\frac{1}{2}[y\pm\sqrt{(y^2-4)}]$.....(3). (2) in (1) gives

$y^2+[4a/(a-1)]y=-[(4a+2)/(a-1)]$.

$\therefore y=[\pm\sqrt{(2a+2)}-2a]/(a-1)$.....(4). (4) in (3) gives

$x=[1/2(a-1)]\{-2a\pm\sqrt{[2a+2]}\pm\sqrt{[10a-2\pm4a\sqrt{(2a+2)}]}\}$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.; and F. M. McGAW, A. M., Professor of Mathemetics in Bordentown Military Institute, Bordentown, N. J.

Expanding and arranging we get, dividing by $a-1$,

$$x^4+\frac{4a}{a-1}x^3+\frac{6a}{a-1}x^2+\frac{4a}{a-1}x+1=0,$$

a reciprocal equation which is easily reduced by dividing by x^2, thus

$$[x^2+(1/x^2)]+\frac{4a}{a-1}[x+(1/x)]+\frac{6a}{a-1}=0,$$

Putting $x+(1/x)=y$, we shall have

$$y^2+\frac{4a}{a-1}y=-\frac{4a+2}{a-1}, \text{ whence } y=-\frac{2a}{a-1}\pm\frac{1}{a-1}\sqrt{2(a+1)},$$

and then

$$x=-\frac{a}{a-1}\pm\frac{1}{2(a-1)}\sqrt{2(a+1)}\pm\frac{1}{2(a-1)}\sqrt{2(2a^2+a+1)\mp4a\sqrt{2(a+1)}}.$$

Also solved by *A. H. BELL*.

GEOMETRY.

84. Proposed by FREDERICK R. HONEY, Ph. B., Instructor in Mathematics in Trinity College, New Haven, Conn.

Find the locus of a point which will trisect all arcs having a common chord.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

In what follows the arc of the circle is assumed.

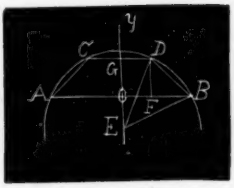
Let O be the origin, $EO=z$, $AB=2c$, $AC=CD=DB=n$.

Then $BF=\frac{1}{2}(2c-n)$, $OF=\frac{1}{2}n$, $DF=\sqrt{n^2-\frac{1}{4}(2c-n)^2}$, $EB^2=c^2+z^2$, $ED^2=EB^2=c^2+z^2$, $EG^2=c^2+z^2-\frac{1}{4}n^2$.

EG = also $EO+DF=z+\frac{1}{2}\sqrt{4n^2-(2c-n)^2}$.

$\therefore\ c^2+z^2-\frac{1}{4}n^2=z^2+z\sqrt{4n^2-(2c-n)^2}+n^2-c^2+cn-\frac{1}{4}n^2$,

or $2c^2-(n^2+cn)=z\sqrt{3n^2-4c^2+4cn}$ .....(1).



Equation to circle is $x^2+(y+z)^2=c^2+z^2$,

or $x^2+y^2+2yz=c^2$. $\therefore\ z=(c^2-x^2-y^2)/2y$ .....(2).

$OF^2+FD^2=x^2+y^2=\frac{1}{4}n^2+n^2-\frac{1}{4}(2c-n)^2$.

$\therefore\ n^2+cn=x^2+y^2+c^2$.

$\therefore\ n=\frac{1}{2}(\sqrt{4x^2+4y^2+5c^2}-c)$ .....(3).

(2) and (3) in (1) gives

$$2c^2-x^2-y^2-c^2=\frac{c^2-x^2-y^2}{2y}\sqrt{4x^2+4y^2-\tfrac{1}{4}(\sqrt{4x^2+4y^2+5c^2}-c)^2}.$$

$\therefore\ 4y^2=3x^2+3y^2-\frac{3}{2}c^2+\frac{1}{2}c\sqrt{4x^2+4y^2+5c^2}$,

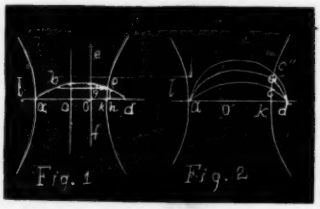
$(2y^2-6x^2+3c^2)^2=c^2(4x^2+4y^2+5c^2)$.

$(y^2-3x^2)^2+2c^2(y^2-5x^2)+c^4=0$.

II. Solution by the PROPOSER.

Let $abcd$--Fig. 1—be any arc, and ad the chord. Bisect ad at o, and draw eof perpendicular to it. We wish to find the locus of a point c whose distance from a given straight line ef is one half the distance from a given point d.

In order to write the equation of this curve, refer it to the coördinate axes ad (axis of X) and ef (axis of Y), intersecting at the origin o.



Let $gc=x$. Therefore, from the definition $cd=2x$. Let $od=D$. $\therefore\ hd=D-x$.

Let $ch=y$. $\therefore\ (2x)^2=y^2+(D-x)^2$ or $4x^2=y^2+D^2-2Dx+x^2$.

$\therefore\ y^2-3x^2+D^2-2Dx=0$....(I). This is the equation of an hyperbola whose center is on the axis of abscissas. In order to determine the position of the center, eliminate the x term, and find the distance from the origin o to a new origin o'. Let E=distance from o to o'. $\therefore\ x=x'+E$. Substituting this value of x in equation (I), $y^2-3(x'+E)^2+D^2-2D(x'+E)=0$; or $y^2-3x'^2-6Ex'-3E^2+D^2-2Dx'-2DE=0$.....(II). In this equation the x' terms should disappear.

$\therefore\ 6Ex'-2Dx'=0$. $\therefore\ E=-\frac{1}{3}D$.

That is, the distance from the origin o to the new origin, or the center of the hyperbola, o', is equal to one third of the distance from o to d; and the minus sign indicates that the measurement should be laid off to the left of the origin o. Substituting this value of E in equation II, and omitting accents, we have $y^2-3x^2+2x-\frac{1}{3}D^2+D^2-2x+\frac{2}{3}(D^2)=0$.

$\therefore\ y^2-3x^2=-\frac{1}{3}(4D^2)$. This is the equation of an hyperbola referred to its center o' as the origin. To write it in the ordinary form, that is, in terms of the transverse and conjugate axes, multiply each term by C; *i. e.* let $\sqrt{C}$=semi-transverse axis. Thus $Cy^2-3Cx^2=-\frac{1}{3}(4CD^2)$.

When in this form the product of the coefficients of the x^2 and y^2 terms should be equal to the remaining term. That is $-3C^2=-\frac{1}{3}(4CD^2)$.....(III). $\therefore\ C=\frac{1}{9}(4D^2)$, and equation III becomes $\frac{1}{9}(4D^2)y^2-\frac{1}{3}(4D^2)x^2=-\frac{1}{27}(16D^4)$.

The semi-transverse axis$=\sqrt{[\frac{1}{9}(4D^2)]}=\frac{1}{3}(2D)$. The semi-conjugate axis $=\sqrt{[\frac{1}{3}(4D^2)]}=2D/\sqrt{3}$.

Since the distance from the center of the curve to either focus is equal to the square root of the sum of the squares of the semi-axes, the distance from o' to either focus$=\sqrt{\{[\frac{1}{9}(4D^2)]+[\frac{1}{3}(5D^2)]\}}=\frac{1}{3}(4D)$. We can therefore make the following construction—Fig. 2. Draw ad the chord of the arc acd. Trisect ad at o' and k. Produce da to l, making $al=ao'=o'k=kd$. With ak as a transverse axis, and l and d as foci, construct the branch of the hyperbola $kcc'c''$, which will intersect all arcs having the common chord ad at c, c', c'', etc., making the arcs cd, $c'd$, $c''d$, respectively, equal to one-third of the arcs acd, $ac'd$, $ac''d$, etc.

CALCULUS.

67. Proposed by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, O.

A man starts to walk at a uniform rate across a draw-bridge just as it begins to move. He walks the full length of the bridge and back, in the same time that it takes the bridge to make a half revolution. How far does he ride, the length of the bridge being 250 feet, and its velocity uniform about a center axis?

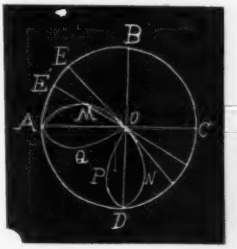
I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

In my solution of problem 45, I deduced the equation, $\rho=r(m-\theta)/m$. Let $m=\frac{1}{4}\pi$. $\therefore\ \rho=(r/\pi)(\pi-4\theta)$ is the equation to the man's path in space.

$$\therefore\ S=\frac{2r}{\pi}\int_0^{\frac{1}{4}\pi}\sqrt{16+(\pi-4\theta)^2}\,d\theta=\tfrac{1}{2}r\sqrt{16+\pi^2}+\frac{8r}{\pi}\left(\frac{\pi+\sqrt{(16+\pi^2)}}{4}\right)$$

$$=\tfrac{1}{2}\sqrt{16+\pi^2}+\frac{1000}{\pi}\log\frac{\pi+\sqrt{(16+\pi^2)}}{4}.\quad\therefore\ S=547.468\text{ feet.}$$

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.



According to the conditions of the problem, A being the starting point of the man, he will be at O, going in the curve AMO, when the draw-bridge has turned through $\angle AOE=45°$; when the latter has turned through $\angle AOB=90°$, the man is in D, having passed through the curve OND; after the bridge has turned through $\angle AOF=135°$, the man is at O again, having moved through the curve DPO, and after the

draw-bridge has made a semi-revolution, he is back at A, having been swept through the curve AQO. Choosing O for the origin of polar coödinates, we have for the curve to the polar equation $r=4R\theta/\pi$, R being the radius of the revolving draw-bridge. The required path of the man is

$$=4\int_0^{\frac{1}{4}\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\frac{16R}{\pi}\int_0^{\frac{1}{4}\pi}\sqrt{\theta^2+1}\,d\theta=\frac{8R}{\pi}[\theta\sqrt{\theta^2+1}+$$

$$\log(\theta+\sqrt{\theta^2+1})\Big]_0^{\frac{1}{4}\pi}=\tfrac{1}{2}R\left[\sqrt{\pi^2+16}+\frac{16}{\pi}\log\left(\frac{\pi+\sqrt{(\pi^2+16)}}{4}\right)\right];$$

for $R=125$ feet, we get this length$=547.45$ feet.

III. Solution by J. M. BANDY, A. M., Civil Engineer for the Roads and Bridges of Guilford County, Greensboro, N. C.

Let M be any position of the man, let $(\rho,\ \theta)$ denote the polar coördinates of M, and let $r=$radius of the bridge. When A shall have revolved through $45°$ the man will be at O, the center; and the limits of θ for $\frac{1}{4}$ the curve are 0 and $\frac{1}{4}\pi$.

Since the man and the motion of the bridge are uniform, AE' and $E'M$ are in a constant ratio, which denote by n. But $AE'=r\theta$, $E'M=r-\rho$, and $n=\frac{1}{4}\pi$. Hence, $r[1-(4/\pi)\theta]$.....(1). By the theory of curves,

$S=\int\left(\rho^2+\frac{d\rho^2}{d\theta^2}\right)^{\frac{1}{2}}d\theta$. From (1), $\rho^2=r^2[1-(4/\pi)\theta]^2$, and $\frac{d\rho^2}{d\theta^2}=\frac{4r^2}{\pi^2}$.

Substituting in formula,

$$S=4\int_0^{\frac{1}{4}\pi}\left[\frac{4r^2}{\pi^2}+\frac{r^2}{\pi^2}(\pi-4\theta)^2\right]^{\frac{1}{2}}d\theta,=\frac{4r}{\pi}\int_0^{\frac{1}{4}\pi}[4+(\pi-4\theta)^2]^{\frac{1}{2}}\,d\theta,$$

$$=\frac{4r}{\pi}\left[\frac{(\pi-4\theta)}{2}\sqrt{4+(\pi-4\theta)^2}+\tfrac{1}{2}\log\left((\pi-4\theta)+\sqrt{4+(\pi-4\theta)^2}\right)\right]_0^{\frac{1}{4}\pi},$$

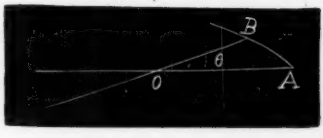
$$=\tfrac{1}{2}r\sqrt{16+\pi^2}+(4r/\pi)\log\left(\frac{\pi+\sqrt{(16+\pi^2)}}{4}\right)$$

$$=1\tfrac{2}{3}\tfrac{5}{0}\sqrt{16+\pi^2}+\frac{1000}{\pi}\log\left(\frac{\pi+\sqrt{(16+\pi^2)}}{4}\right).$$

$S=547.45$ feet.

IV. Solution by C. W. M. BLACK, A. M., Professor of Mathematics, Wesleyan Academy, Wilbraham, Mass.

Let $a=$number of complete revolutions made while he walks the full length. $b=2OA=$length of draw-bridge. Let AB be curve traversed. $OB=\rho$, $\angle AOB=\theta$.



Then $\frac{1}{2}b-\rho : b=\theta : 2a\pi$.

$$\rho=b(\tfrac{1}{2}-\frac{\theta}{2a\pi}).\qquad \frac{d\rho}{d\theta}=-\frac{b}{2a\pi}.$$

s (distrance traversed in passing to end of bridge)

$$=\int_0^{2a\pi}\left[b^2\left(\tfrac{1}{2}-\frac{\theta}{2a\pi}\right)^2+\frac{b^2}{4a^2\pi^2}\right]^{\frac{1}{2}}d\theta=(\tfrac{1}{2}b)\sqrt{1+a^2\pi^2}-$$

$$\frac{b}{2a\pi}\log_e(\sqrt{1+a^2\pi^2}-a\pi).\quad b=250.\quad a=\tfrac{1}{4}.$$

$$\therefore\ 2s=250[1+(\tfrac{1}{16}\pi^2)]-(1000/\pi)\log_e\{\sqrt{[1+(\tfrac{1}{16}\pi^2)]}-\tfrac{1}{4}\pi\}=547.6+\text{ feet.}$$

[See also solutions of problems 41, 45, and 50, published in previous numbers of MONTHLY. EDITOR.]

MECHANICS.

56. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics in Manual Training School, Philadelphia, Pa.

"Hey-diddle-diddle, the cat and the fiddle,
The cow jumped over the moon."

Taking the weight of the cow to be 600 pounds, the initial resistance of the air to be 100 pounds and varying as the square of the velocity, find the initial and final velocities, and the times of rising and falling.

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let R=radius of earth=20924640 feet,

$h=60.27R=1261128052.8$ feet=distance to moon,

$g=32.2$ feet, $W=600$ pounds, $\mu v^2=100$ pounds$=\frac{1}{6}W$,

t=time of ascent; t_1=time of descent, v=initial velocity, v_1=final velocity. Then $\mu=W/6v^2$; $1/k=\sqrt{(W/\mu)}=v\sqrt{6}$. $\therefore\ k=1/v\sqrt{6}$.

$h=(1/2gk^2)\log(1+v^2k^2)$, $\therefore\ t=(1/g)\tan^{-1}vk$,

$t_1=(1/gk)\log[\sqrt{(1+v^2k^2)}+vk]$, $v_1=v/\sqrt{(1+v^2k^2)}$.

$\therefore\ (1+v^2k^2)=\frac{7}{6}$, $vk=1/\sqrt{6}$.

$\therefore\ v^2=gh/3\log(\frac{7}{6})$, $v=449657$ feet$=85.16$ miles per second.

$v_1=(\frac{6}{7})v=73$ miles per second.

$t=[v\sqrt{(6)}/g]\tan^{-1}(1/\sqrt{6})=13258.2$ seconds$=3$ hours, 40 minutes, 58.2 seconds.

$t_1=[v\sqrt{(6)}/g]\log\{[\sqrt{(7)}+1]/[\sqrt{6}]\}=13603.7$ seconds$=3$ hours, 46 minutes, 43.7 seconds.

In the above we have considered the resisting medium as extending to the moon.

57. Proposed by J. C. NAGLE, A. M., M. C. E., Professor of Civil Engineering, Agricultural and Mechanical College of Texas, College Station, Texas.

Over the intersection of two inclined planes slides a cord of uniform mass throughout its length. Find the equation of the path described by its center of gravity.

[No solution of this problem has been received. EDITOR.]

58. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

An endless uniform chain is hung over two small smooth pegs in the same horizontal line. Show that, when it is in a position of equilibrium, the ratio of the distance between the vertices of the two catenaries to half the length of the chain is the tangent of half the angle of inclination of the portions near the pegs. [From *Routh's Analytical Statics. Mathematicl Tripos, 1855.*]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let h_1, h_2 be the vertical distances of the vertices of the two catenaries below the line ($=2a$) joining the pegs ; then using the usual notation of the theory of the catenary, the origin being the middle of $2a$,

$$h_1+c_1-y=\tfrac{1}{2}c_1(e^{x/c_1}+e^{-(x/c_1)})\ldots\ldots(1),$$

$$s_1=\tfrac{1}{2}c_1(e^{x/c_1}-e^{-(x/c_1)})\ldots\ldots(2),$$

with like equations for the second curve.

From (1) and (2), α_1 being the inclination of the curve at the peg,

$$(dy/dx)=\tfrac{1}{2}(e^{x/c_1}-e^{-(x/c_1)})=\tan\alpha_1\ldots\ldots(3).$$

$$(ds_1/dx)=\tfrac{1}{2}(e^{x/c_1}+e^{-(x/c_1)})=\sec\alpha_1\ldots\ldots(4).$$

$$\therefore \sec\alpha_1+\tan\alpha_1=e^{a/c_1}\ldots\ldots(5).$$

$y=0$ in (1) gives $h_1+c_1=\tfrac{1}{2}c_1(e^{a/c_1}+e^{-(a/c_1)})=s_1/\sin\alpha_1\ldots\ldots(6)$,

by aid of (5), and by the usual theory, $c_1=s_1\cot\alpha_1\ldots\ldots(7)$, and then from (6) and (7),

$h_1=s_1\tan\tfrac{1}{2}\alpha_1\ldots(8)$. Similarly, $h_2=s_2\tan\tfrac{1}{2}\alpha_2\ldots(9)$, and $h_2+c_2=s_2/\sin\alpha_2\ldots(10)$.

For equilibrium, the right members of (6) and (10) are equal, since they measure the tensions in the two parts of the chain at either peg.

$$\therefore \frac{s_1}{s_2}=\frac{\sin\alpha_1}{\sin\alpha_2}=\frac{\cot\tfrac{1}{2}\alpha_2+\tan\tfrac{1}{2}\alpha_2}{\cot\tfrac{1}{2}\alpha_1+\tan\tfrac{1}{2}\alpha_1}\ldots\ldots(11).$$

Then finally,

$$\frac{h_1-h_2}{s_1+s_1}=\frac{(s_1/s_2)\tan\tfrac{1}{2}\alpha_1-\tan\tfrac{1}{2}\alpha_2}{(s_1/s_2+1}\ldots\ldots(12).$$

(11) in (12) gives

$$\frac{h_1-h_2}{s_1+s_1}=\tan\tfrac{1}{2}(\alpha_1-\alpha_2)\ldots\ldots(13).$$

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O. University, Miss., and HENRY HEATON, M. Sc., Atlantic, Iowa.

Let $2l$ be the length of the lower catenary, c its parameter, ψ the angle which its tangent at the right-hand peg makes with its directrix ; and let the same letters, accented, refer, similarly, to the upper catenary.

Then $\tan\psi=l/c$ and $\tan\psi'=l'/c'$, from which

$$\tan(\psi-\psi')=\frac{c'l-cl'}{cc'+ll'};\text{ and }\tan\tfrac{1}{2}(\psi-\psi')=\frac{\sqrt{(c^2+l^2)(c'^2+l'^2)}-cc'-ll'}{c'l-cl'}.$$

The tension in the chain at any point being equal to the weight of as much of it as would reach from that point to the directrix, and the resultant tensions in the two catenaries at the pegs being the same, it follows that the two catenaries have a common directrix. Also, the square of the ordinate of any point equals the sum of the squares of the parameter and the length of the arc between the vertex and that point. From these facts, $l^2+c^2=l'^2+c'^2$. Substituting,

$$\tan\tfrac{1}{2}(\psi-\psi')=\frac{c^2+l^2-cc'-ll'}{c'l-cl'}=\frac{(c-c')c+(l-l')l}{c'l-cl'}.$$

Remembering that $(c-c')(c+c')=(l'-l)(l+l')$, this may be reduced to $(c'-c)/(l+l')$.

But $c'-c$ is the distance between the vertices of the two catenaries, $l+l'$ is half the length of the chain, and $\psi-\psi'$ is the inclination of the portions near the peg. Hence the proposition is established.

Also solved by *G. B. M. ZERR.*

DIOPHANTINE ANALYSIS.

57. Proposed by JOSIAH H. DRUMMOND, LL. D., Counselor at Law, Portland, Maine.

Each of *five* of the digits may be the terminal figure of a perfect integral square. Each of *eighteen* combinations of two digits may be the *two* terminal figures of an integral square. Each of *one hundred and nineteen* combinations of three digits may be the *three* terminal figures of an integral square. *Under these conditions*, what is the greatest number of arrangements of the nine digits, all taken together, whose three terminal figures shall be those of a square number?

Solution by the PROPOSER.

The first two conditions are embraced in the third and must be disregarded: the terminal figure of the last three must be one of five digits; and the two terminal figures must be one of the eighteen combinations of two digits. There are 119 combinations of three digits that may be terminal figures: with each one of these combinations, within the terms of the question, the other six digits may be used as many times as there are combinations in the six taken together, that is $2\times3\times4\times5\times6=720$. Hence $119\times720=85{,}680$, the number required.

59. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Find the sum of the mth power of all the numbers less than P and prime to it, and then by substitution find the sum when $m=1, 2, 3, 4, 5$.

Solution by the PROPOSER.

The sum of the mth powers of all numbers less than P and not prime to it is given by the sum of

$$a^m+(2a)^m+(3a)^m+\ldots\ldots+[(P/a).a]^m$$
$$+b^m+(2b)^m+(3b)^m+\ldots\ldots+[(P/b).b]^m$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$-(ab)^m-(2ab)^m-(3ab)^m-\ldots\ldots-[(P/ab).ab]^m$$
$$-(bc)^m-(2bc)^m-(3bc)^m-\ldots\ldots-[(P/bc).bc]^m$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+(abc)^m+(2abc)^m+(3abc)^m+\ldots\ldots+[(P/abc).abc]^m$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$=a^mS_{P/a}+b^mS_{P/b}+\ldots\ldots-(ab)^mS_{P/ab}-(bc)^mS_{P/bc}-\ldots\ldots+(abc)^mS_{P/abc}+\ldots\ldots$$

$\therefore$ The sum of the mth powers of all the integers less than P and prime to it is

$$S_P-a^mS_{P/a}+b^mS_{P/b}-\ldots+(ab)^mS_{P/ab}+(bc)^mS_{P/bc}+\ldots-(abc)^mS_{P/abc}-\ldots\ldots(1).$$

But $$S_P=\frac{P^{m+1}}{m+1}+\tfrac{1}{2}P^m+B_1\frac{m}{2\,!}P^{m-1}+B_3\frac{m(m-1)(m-2)}{4\,!}P^{m-3}$$
$$+B_5\frac{m(m-1)(m-2)(m-3)(m-4)}{6\,!}P^{m-5}+\ldots\ldots$$

$$a^mS_{P/a}=\frac{P^{m+1}}{m+1}.(1/a)+\tfrac{1}{2}P^m+B_1\frac{m}{2\,!}P^{m-1}a+B_3\frac{m(m-1)(m-2)}{4\,!}P^{m-3}a^3$$
$$+B_5\frac{m(m-1)(m-2)(m-3)(m-4)}{6\,!}P^{m-5}a^5+\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Where $B_1=\frac{1}{6}$, $B_3=\frac{1}{30}$, $B_5=\frac{1}{42}$, $B_7=\frac{1}{30}$, etc., Bernoulli's numbers. Substituting the values of S_P, $a^mS_{P/a}$, $b^mS_{P/b}$, $(ab)^mS_{P/ab}$, etc., in (1) and letting S_m=the sum, we get

$$S_m=\frac{P^{m+1}}{m+1}\left(1-(1/a)-(1/b)-(1/c)-\ldots\ldots+(1/ab)+\ldots\ldots\right)$$
$$+\tfrac{1}{2}P^m\left(1-n+\frac{n(n-1)}{2\,!}-\ldots\ldots\right)+B_1\frac{m}{2\,!}P^{m-1}\{1-a-b-c-\ldots\ldots+ab+\ldots\ldots\}$$
$$-B_3\frac{m(m-1)(m-2)}{4\,!}P^{m-3}\{1-a^3-b^3-c^3-\ldots\ldots+a^3b^3+\ldots\ldots\}$$
$$+B_5\frac{m(m-1)(m-2)(m-3)(m-4)}{6\,!}P^{m-5}\{1-a^5-b^5-c^5-\ldots\ldots+a^5b^5+\ldots\ldots\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Where n is the number of primes in P. $\therefore$ The coefficient of P^m is zero.

$$\therefore S_m=\frac{P^{m+1}}{m+1}[1-(1/a)][1-(1/b)][1-(1/c)]\ldots.$$

$$+B_1\frac{m}{2\,!}P^{m-1}(1-a)(1-b)(1-c)\ldots.$$

$$-B_3\frac{m(m-1)(m-2)}{4\,!}P^{m-3}(1-a^3)(1-b^3)(1-c^3)\ldots.$$

$$+B_5\frac{m(m-1)(m-2)(m-3)(m-4)}{6\,!}P^{m-5}(1-a^5)(1-b^5)(1-c^5)\ldots.$$

$$-\ldots\ldots+\ldots\ldots-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

When $m=1$, $S_1=\frac{1}{2}P^2[1-(1/a)][1-(1/b)][1-(1/c)]\ldots.$

When $m=2$, $S_2=\frac{1}{3}P^3[1-(1/a)][1-(1/b)][1-(1/c)]..+\frac{1}{6}P(1-a)(1-b)(1-c)..$

When $m=3$, $S_3=\frac{1}{4}P^4[1-(1/a)][1-(1/b)][1-(1/c)]..+\frac{1}{4}P^2(1-a)(1-b)(1-c)..$

When $m=4$, $S_4=\frac{1}{5}P^5[1-(1/a)][1-(1/b)][1-(1/c)]..+\frac{1}{3}P^3(1-a)(1-b)(1-c)..$

$$-\tfrac{1}{3}\tfrac{1}{0}P(1-a^3)(1-b^3)(1-c^3)\ldots.$$

When $m=5$, $S_5=\frac{1}{6}P^6[1-(1/a)][1-(1/b)][1-(1/c)]..+\tfrac{5}{1}\tfrac{5}{2}P^4(1-a)(1-b)(1-c)..$

$$-\tfrac{1}{1}\tfrac{1}{2}P^2(1-a^3)(1-b^3)(1-c^3)\ldots.$$

When $m=6$, $S_5=\frac{1}{7}P^7[1-(1/a)][1-(1/b)][1-(1/c)]..+\frac{1}{2}P^3(1-a)(1-b)(1-c)..$

$$-\tfrac{1}{6}P^3(1-a^3)(1-b^3)(1-c^3)\ldots.\tfrac{1}{4}\tfrac{1}{2}P(1-a^5)(1-b^5)(1-c^5)\ldots.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

83. Proposed by J. MARCUS BOORMAN, Consultative Mechanician, Counselor at Law, Inventor, Etc., Woodmere, Long Island, N. Y.

Solve $x^2+y=8\ldots\ldots(1)$; $y^2+x=69\ldots\ldots(2)$, true to four decimals.

84. Proposed by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, O.

On the present electoral basis, if all the electoral votes of each State are cast solid for one or the other of two presidential candidates, how many conbinations of States are possible for a total of 273 votes for the winning candidate?

*** Solutions of these problems should be sent to J. M. Colaw, not later than May 10.

GEOMETRY.

91. Proposed by LEONARD E. DICKSON, Ph. D., Instructor in Mathematics in the University of California, Berkeley, Cal.

If a point A remain fixed while a point B moves along a given straight line, prove that the locus of the vertex C of the triangle ABC, similar to a given triangle and lying always on the same side of AB, is a straight line. Verify geometrically for the case in which the angles at A and C remain equal.

92. Proposed by JOSIAH H. DRUMMOND, LL. D., Counselor at Law, Portland, Maine.

Let $ABCD$ be a quadrilateral inscribed in a circle. Draw the diagonals AC and BD. Show that $AB.BC : DC.AD = BD : AC$. [From a note in *Young's Geometry*, edition of 1830.]

93. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

While surveying in a level field I notice a mountain behind a hill. Wishing to know the height of each I take the angles of elevation of the tops of both and find them to be $\beta=45°$, $\delta=40°$, I then measure a straight line $a=400$ feet and find the angles of elevation of the tops to be $\gamma=42°$, $\mu=38°$. After measuring $b=300$ feet more in the same straight line I find the elevations to be $\lambda=40°$, $\nu=36°$. Find the height of each.

*** Solutions of these problems should be sent to B. F. Finkel, not later than May 10.

CALCULUS.

73. Proposed by MOSES COBB STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Ind.

$$\text{Solve} \int_0^{\frac{1}{4}\pi} \log(1-\tan x)\,dx$$

74. Proposed by EDWARD R. ROBBINS, A. B., Mathematical Master in the Lawrenceville School, Lawrenceville, N. J.

A circular ring, whose radii are a and b, is cut by a plane making the area of the section (or sections) a maximum. Required the position of the plane, and the nature and area of the section (or sections).

*** Solutions of these problems should be sent to J. M. Colaw, not later than May 10.

DIOPHANTINE ANALYSIS.

64. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find two cubic proper fractions whose product is a square proper fraction. Can a *general* solution be made?

65. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Find (1) four consecutive numbers whose sum is a square, and (2) four consecutive numbers the sum of whose squares is a square.

*** Solutions of these problems should be sent to J. M. Colaw, not later than May 10.

MISCELLANEOUS.

60. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

A tube of uniform cross section, small compared with its length, is bent into the form of a cycloid, its open ends lying at the cusps, and this cycloid is placed with its axis vertical and its vertex downwards. Equal quantities of fluids of specific gravity σ_1 and σ_2 are

poured in at the two cusps, the quantity of each being such as would fill a length of the tube equal to its axis a. If the fluids do not mix, find the distance x_1, x_2 of the upper levels of the fluids from the vertex measured along the cycloidal arc. [From *Proctor's Geometry of the Cycloid.*]

61. Proposed by F. M. SHIELDS, County Surveyor, Coopwood, Miss.

Of three chronometers, A, B, and C, A keeps true time; B gains 5 minutes and 15¾ seconds a day by true time, and C loses 7 minutes and 15 3-7 seconds a day by true time. The hands of all three watches are set at 12 noon on a certain day. What is the time by the true watch, A, on the *fifth* day after time when the hands of the fast watch, B, point to 12, and what is the time by the true watch, A, on the *tenth* day after time when the hands of the slow watch, C, point to 12?

*** Solutions of these problems should be sent to J. M. Colaw, not later than May 10.

PERIODICALS.

The Open Court. A Monthly Magazine. Devoted to Science of Religion, the Religion of Science, and the extension of the Religious Parliament Idea. Edited by Dr. Paul Carus; Assistant Editor, T. J. McCormack and Associate Editors E. C. Hegeler and Mary Carus. Price, $1.00 per year in advance. Single copies, 10 cents.

The April *Open Court* presents its readers as a frontispiece with a unique specimen of Japanese art in the shape of a brilliantly colored poster called *The Pure Land* or *Western Paradise*, by a famous Tokyo artist, Mishima. The posters were made especially for the *Open Court*, and the sheets imported from Japan. The conception is delicate, and the tints little short of exquisite.

In the leading article Dr. Woods Hutchinson, of Buffalo discourses upon Courage the Chief Virtue, which is opposed by the author to meekness and submissiveness usually taught, and contrasts the sublime fortitude of Christ to the cowardice of many interpretations of his religion. Dr. Moncure D. Conway continues his series of articles on Solomonic Literature, giving in the present number the mythological interpretation of the traditions connected with the wives of Solomon. The illustraded article of the number is by Dr. Paul Carus on the Human Heart as Mirrored in Religious Art. Old wood cuts are reproduced, portraying the various conceptions of the soul as the vehicle of good and evil, while representations of modern ideas are also given.

Captain Pfoundes of Japan writes entertainingly of the present situation in China, and his article is accompanied by a handsome half tone illustration of the Seven Sages of the Bamboo-Grove. Mr. Sandison of Glasgow reports the Gifford Lectures now being delivered in Scotland by Dr. Bruce; and Mr. Edmund Noble of Boston gives us Some Parallels Between Theology and Science. Prof. I. W. Howerth of the University of Chicago treats of the crying problems of social life. Poems, with a new hymn by Dr. Paul Carus, and a number of book reviews complete the number.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

The Cuban crisis naturally demands more space in the editorial department of the *American Montly Review of Reviews* than any other single topic. The whole matter is reviewed in the light of the latest and most authentic information received up to the time of going to press. The *Review* is convinced that the country desires and will demand intervention in Cuba, that the real question at issue is the relief of Cuba, not the settlement of the *Maine* incident, and that Spain's final withdrawal from the Western hemisphere will be the only satisfactory termination of the present trouble.

The Monist. A Quarterly Magazine devoted to the Philosophy of Science. Edited by Dr. Paul Carus; T. J. McCormack, Assistant Editor; E. C. Hegeler, and Mary Carus, Associate Editors. Price, $2.00 per year in advance. Single number, 50 cents. The Open Court Publishing Co., Chicago, Ill.

Prof. Ferdinand Hüppe, the well known Professor of Hygiene in the University of Prague, contributes an interesting and important article to the April *Monist* on the Causes of Infectious Diseases. Prof. Hüppe is a bacteriologist of the modern school but nevertheless opposes the main doctrines of Koch, Pasteur and Virchow, and few will dissent from the reasonableness of the position he takes, which harmonizes the facts of the new theories with the established principles of the old. Both physicians and laymen will be interested in Prof. Hüppe's presentation of modern bacteriology.

In the same number, the famous Italian criminologist, Prof. Cesare Lombroso seeks to establish his favorite theory of the degeneracy of genius by considering certain regressive phenomena in evolution. Dr. Woods Hutchinson writes passionately and with rare ability upon Lebenslust, or the joy of life. A distinguished English lady, E. E. Constance Jones, discusses An Aspect of Attention. Prof. John Dewey, of the University of Chicago, discusses ethics in the light of evolution. And, finally, the editor, Dr. Paul Carus, in a long article on the Unmateriality of Soul and God, seeks to lay a firm foundation for correct views of these momentous questions. The number concludes with entertaining Literary Correspondence from Europe, and the usual number of Book Reviews in the field of science, philosophy and religion.

SOME ERRATA IN FEBRUARY NUMBER.

Page 42, line 34, for "defective" read defect.
Page 43, line 12, for "$-xy=6$" read $-xy=-6$.
Page 44, line 17, for last "$\pm_1$" read $\mp_1$; line 30, omit "(" before xy.
Page 47, line 18, for "PN" read DN; line 19, supply "(" before $FG+NE$).
Page 48, line 7, for "$12\pi n+\pi n^2/1200=100$" read $12\pi n+\pi n^2/100=1200$.
Page 52, line 28, for "$3(6^2-1^1)$" read $3(6^2-1^2)$.
Page 53, line 3, after 208, insert 608; line 33, for "13" read 15.
Page 54, line 3, for "363" read 368.
Page 59, line 21, for "$7m^2n$" read $7m^3n$.
Page 60, line 1 of problem 70, for "π/n" read $\pi/2n$; and in last line, for "$\pi/2l$" read $\pi/2n$.
Page 61, problem 63, for "x^2" read x^3, and for "105498" read 105489.
Page 66, line 13 from bottom, for "Page 22" read Page 17.